Chapter 1 : Systems of Linear Equations

Linear equations in n variables

leading coefficient, variable — highest power = 1, produces straight line

$$\boxed{a_1} x_1 - a_2 \boxed{x_2} + a_3 x_3 + \dots + a_n x_n = \boxed{b}$$

coefficient — variable — constant, a real number

Solutions and solution sets of a linear eqn

↳ a sequence of n real numbers $s_1, s_2, s_3, \dots, s_n$ that satisfy the eqn when the value is substituted into eqn

$x_1 = s_1,\ x_2 = s_2,\ x_3 = s_3,\ \dots,\ x_n = s_n$

↳ set of all solutions = $\boxed{\text{solution set}}$

can be represented using **Parametric Representation**

eg: Solve the linear eqn $3x + 2y - z = 3$

① Choose y and z to be free variables

$3x = 3 - 2y + z$

$x = 1 - \frac{2}{3}y + \frac{1}{3}z$

② letting $y = s$, $z = t$ to obtain parametric rep

$x = 1 - \frac{2}{3}s + \frac{1}{3}t$, where $y = s$ and $z = t$

particular solutions : $x = 1, y = 0, z = 0$ and $x = 1, y = 1, z = 2$

Systems of Linear Equations / linear system

↳ sets of linear equations with the same set of variables

Number of solutions of a system of linear eqns

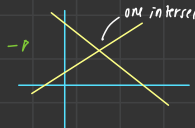
① One solution - consistent, independent

$x + y = 3$

$x - y = -1$ →

$x = 1,\ y = 2$

one intersection



② Infinitely many - consistent, dependant

$x + y = 3$

$2x + 2y = 6$

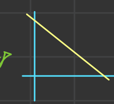
the 2nd eqn is (1st eqn) × 2, there are infinitely many solutions.

↳ the solution set is represented using parametric rep

$x = 3 - t$, $y = t$, t is any real number

it's essentially having only 1 eqn for 2 variables

→



no constriction of value to put into eqn

↓

infinitely many value

↓

infinitely many solution

③ No solution - inconsistent, independant

$x + y = 3$

$x + y = 1$

no solution, x + y can't be 3 and 1 simultaneously →

Row Echelon Form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

All entries below the leading entries are "0"

Types of solution of $Ax = b$

① unique solution

$$\left[\begin{array}{ccc|c} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \end{array}\right]$$

② No solution

$$\left[\begin{array}{ccc|c} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & 0 & * \end{array}\right]$$

③ Infinitely many solution

$$\left[\begin{array}{ccc|c} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

— no. of variables > no. of eqns

↓

not enough information to constraint the variables

↓

many number that can satisfy the sln

free variable

↳ infinitely many sln sets

Solving Linear Equations Via Matrices Using:

① Gaussian Elimination

eg:
$$x - 2y + 3z = 9$$
$$-x + 3y = -4$$
$$2x - 5y + 5z = 17$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} \quad R_2 + R_1 \to R_2$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \quad R_3 + (-2)R_1 \to R_3$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad R_3 + R_2 \to R_3$$

Row-echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \left(\tfrac{1}{2}\right)R_3 \to R_3$$

Using back-substitution,

$z = 2 \to ①$

$$x - 2y = 9 - 3(2)$$
$$x = 3 + 2y$$

$x = 3 + 2y \to ②$

$$2(3+2y) - 5y + 5(2) = 17$$
$$y = -1$$

$x = 1, \; y = -1, \; z = 2$

② Gauss-Jordan Elimination

Continue until reduced-row echelon form

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 + 2R_2 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_2 + (-3)R_3 \to R_2$$

Reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_1 + (-9)R_3 \to R_1$$

$\therefore \; x = 1$

$y = -1$

$z = 2$

③ Inverse Matrix

$$Ax = B$$
$$x = A^{-1}B$$

Homogenous System

$a_{11} x_1 + \dots a_{1n} x_n = 0$

$\vdots$

$a_{m1} x_1 + \dots + a_{mn} + x_n = 0$

each of the constant terms is 0

eg: $x_1 - x_2 + 3x_3 = 0$
$2x_1 + x_2 + 3x_3 = 0$

} if eqns < variables = infinitely many solutions

using Gauss-Jordan Elimination,

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

using parametric rep, $t = x_3$

solution set: $x_1 = -2t$, $x_2 = t$, $x_3 = t$

$\therefore$ System has many solutions, one of which is the $\boxed{\text{trivial solution}}$ $(t=0)$

↳ each eqns are satisfied

Chapter 2 : Matrices

Matrix Operations

① Addition , Substraction

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} (-1+1) & (2+3) \\ (0+(-1)) & (1+2) \end{bmatrix}$$

② Scalar multiplication

$$3\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3(-1) & 3(2) \\ 3(0) & 3(1) \end{bmatrix}$$

③ Matrix multiplication

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\underbrace{A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}}_{3\times 2}, \quad \underbrace{B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}}_{2\times 2}$$

if same, can multiply

$3\times 2 \ \times \ 2\times 2$

answer'll be 3×2

$c_{11} = (-1)(-3) + (3)(-4)$

$c_{12} = (-1)(3) + (3)(1)$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

Applications of Matrix Multiplication

① Representing a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

can be written as

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

or

$$\boxed{x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

eg: Solve the matrix eqn $Ax = 0$ where

coefficient

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and } 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 + 3x_2 - 2x_3 &= 0 \end{aligned}$$

$\downarrow$ Gauss-Jordan elimination

augmented matrix

$$\begin{bmatrix} 1 & 0 & -1/7 & 0 \\ 0 & 1 & -4/7 & 0 \end{bmatrix}$$

$\therefore x_3 = 7t$, solution set = $x_1 = t, x_2 = 4t, x_3 = 7t$

$\downarrow$ in matrix form

$$x = t\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

Properties of Matrix

1. $A+B = B+A$ - Commutative
2. $A+(B+C) = (A+B)+C$ - associative of addition
3. $(xy)A = x(yA)$ - associative of multiplication
4. $1A = A$ - multiplicative
5. $x(A+B) = xA + xB$ - distributive
6. $(c+d)A = cA + dA$ - distributive
7. $A + \underbrace{O}_{\text{zero matrix}} = A$
8. $A + (-A) = O$
9. if $xA = O$, then $c=O$ or $A=O$

properties of zero matrix

10. $A(BC) = (AB)C$ - associative of multiplication
11. $A(B+C) = AB + AC$ - distributive
12. $x(AB) = (xA)B = A(xB)$
13. if A is a matrix of size $m \times n$, then
    - $A \underbrace{I_n}_{\text{identity}} = A$ → $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
    - $A I_m = A$

Properties of $\boxed{\text{transpose}}$ matrix

rows → columns

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(xA)^T = x(A^T)$
4. $(AB)^T = B^T A^T$

eg: $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

Finding inverse of matrix

① $AA^{-1} = I$

eg: Find inverse matrix of $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$

by using Gauss-Jordan elimination,

$$[A \quad I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_2 + (-1)R_1 \to R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \quad R_3 + 6R_1 \to R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \quad R_3 + 4R_2 \to R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad (-1)R_3 \to R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_2 + R_3 \to R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \quad R_1 + R_2 \to R_1$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

- if $A = 2 \times 2$ matrix
- A^{-1} exist only if $ad - bc \neq 0$

② $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Properties of inverse matrix

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = \underbrace{A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \text{ times}} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c} A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$
5. $(AB)^{-1} = B^{-1} \cdot A^{-1}$
6. If $AC = BC$, then $A = B$

Elementary Matrix

↳ when it can be obtained from the identity matrix by a single operation $I = 3\times 3$

↳ if $\boxed{3}\times 2$

eg: Find a sequence of elementary matrices that can be used to write matrix A in row-echelon form

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \quad E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} \quad R_3 + (-2)R_1 \to R_3 \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad (\tfrac{1}{2})R_3 \to R_3 \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tfrac{1}{2} \end{bmatrix}$$

$$A E_1 E_2 E_3 = B$$

Writing a Matrix as the Product of Elementary Matrices

eg: $A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$

Find out sequence of Elementary Matrices to rewrite A in reduced row-echelon form

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad (-1)R_1 \to R_1 \quad E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad R_2 + (-3)R_1 \to R_2 \quad E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \tfrac{1}{2}R_2 \to R_2 \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \tfrac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_1 + (-2)R_2 \to R_1 \quad E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

∴ since $A E_1 E_2 E_3 E_4 = I$

$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$

use $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The LU-Factorization

eg: solve the linear system

$$u_1 - 3u_2 = -5$$
$$u_2 + 3u_3 = -1$$
$$2u_1 - 10u_2 + 2u_3 = -20$$

① Find the LU Factorization

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} \xrightarrow{R_3 + (-2)R_1 \to R_3} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & -4 & 2 \end{bmatrix} \xrightarrow{R_3 + 4R_2 \to R_3} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$A E_1 E_2 = U$$
$$A = U E_1^{-1} E_2^{-1}$$
$$= U L$$
$$\therefore E_1^{-1} E_2^{-1} = L$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix}$$

②

$$\begin{array}{cccc} U & L & u & B \\ \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} & \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = & \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \end{array}$$

let $y = Ux$

$$\begin{array}{ccc} L & y & b \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} & \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = & \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \end{array}$$

$y_1 = -5, \quad y_2 = -1, \quad y_3 = -14$

$$\begin{array}{ccc} U & u & y \\ \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} & \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = & \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix} \end{array}$$

$$u_2 + 3(-1) = -1$$
$$u_2 = 2$$

$$u_1 = -5 + 3(2)$$
$$= 1$$

Chapter 3 : Determinants

↳ a scalar value that can be calculated from the elements of matrix

Calculating Determinant

① 2×2 matrix

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

③ Triangular matrix

$$A = \begin{bmatrix} 2 & 0 & c \\ 1 & 3 & 0 \\ 5 & 6 & 4 \end{bmatrix}, \quad |A| = (2)(3)(4) = 24$$

product of entries on the main diagonal

② n > 2 matrix

eg: $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$ → we'll use the formula – $|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$ ← for row expansion

– $|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$ ← for column expansion

in this case, we chose 1st row expansion

① Find the minor (M_{ij}) for a row

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1(1) - 2(1) = -1$$

minor of $_{11}$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 3(1) - 4(2) = -5$$

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}, \quad M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 3(0) - 4(-1) = 4$$

② Find the cofactors — $(-1)^{i+j} M_{ij}$

using sign pattern, $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$C_{11} = -1, \; C_{12} = 5, \; C_{13} = 4$

③ Use formula, $|A| = a_{i1} C_{i1} + \ldots + a_{in} C_{in}$

$$|A| = 0(-1) + 2(5) + 1(4) = 14$$

eg: $A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}$

since the 3rd column contains 3 0s, we'll choose it

$$|A| = 3C_{13} + \underbrace{0C_{23} + 0C_{33} + 0C_{43}}_{\text{since these'll result in 0, we can ignore}}$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

we choose to expand 2nd row since there's 0

$$C_{13} = 0(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + 2(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + 3(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$C_{13} = 13$

$|A| = 3C_{13} = 3(13) = 39$ #

Properties of determinants

① B = Row interchange on A, $|B| = -|A|$

② B = Row addition on A, $|B| = |A|$

③ B = Row multiplication by a constant c on A, $|B| = c|A|$

④ B = Elementary column operations on A, $|B| = |A|$

⑤ $|AB| = |A||B|$

⑥ $|cA| = c|A|$

⑦ $|A^{-1}| = 1/|A|$

⑧ $|A| = |A^T|$

eg: $A = \begin{bmatrix} -1 & 2 & 2 \\ 3 & -6 & 4 \\ 5 & -10 & -3 \end{bmatrix} \xrightarrow{C_2 + (-2)C_1 \to C_2} \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & 4 \\ 5 & 0 & -3 \end{bmatrix}$

$|A| = 0C_{12} + 0C_{22} + 0C_{32} = 0$

easier to calculate cofactor

Adjoint of a matrix

$$\text{Cofactor} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Transpose of cofactors

$$\text{Adjoint} = \text{Cofactor}^T = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

Finding inverse using adjoint

$$A^{-1} = \frac{1}{|A|}\,\text{adj}|A|$$

Cramer's Rule (solving system of a linear eqn with determinant)

$$x = \frac{D_x}{D} \quad y = \frac{D_y}{D} \quad z = \frac{D_z}{D}$$

eg:
$$-x + 2y - 3z = 1$$
$$2x \quad + 2 = 0$$
$$3x - 4y + 4z = 2$$

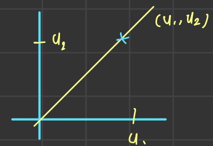
$$|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 2 \\ 3 & -4 & 4 \end{vmatrix} = 10$$

to solve x, the x-column is removed, then proceed with finding determinant

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{1(-1)^5\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} + \overbrace{0(-1)^{2+2}\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -4 & 4 \end{vmatrix}}^{\text{this'll result in 0, that's why we choose row-2}}}{10} = \frac{4}{5}$$

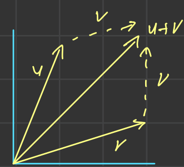
Chapter 4: Vector Spaces

Vector $u = (u_1, u_2)$ =



Vector addition

$u+v = (u_1, u_2) + (v_1, v_2) = (u_1+v_1,\ u_2+v_2) =$



Vector Operations

① $u+v = v+u$
② $(u+v)+w = u+(v+w)$
③ $u+0 = u$
④ $u+(-u) = 0$
⑤ $c(u+v) = cu+cv$
⑥ $(c+d)u = cu+du$
⑦ $c(du) = d(cu)$
⑧ $1u = u$

Vectors in R^{n} dimension

↳ $R^2 = (x, y)$, $R^3 = (x, y, z)$ ...

Linear combinations of vectors

↳ writing one vector as the sum of scalar multiples of other vectors, $u = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$

eg: $u = (-1, -2, -2)$, $u = (0, 1, 4)$, $v = (-1, 1, 2)$, $w = (3, 1, 2)$
Find scalars a, b, c such that $u = au + bv + wc$

$(-1, -2, -2) = a(0, 1, 4) + b(-1, 1, 2) + c(3, 1, 2)$
$= (0, a, 4a) + (-b, b, 2b) + (3c, c, 2c)$
$= (-b+3c,\ a+b+c,\ 4a+2b+2c)$

make comparison,
$-1 = -b+3c$
$-2 = a+b+c$
$-2 = 4a+2b+2c$
} $a = 1, b = -2, c = -1$

$\therefore u = u - 2v - w$

eg: show set of all 2×3 matrices with operation of matrix addition and scalar multiplication is a vector space
↳ since $A+B = 2\times3$ matrices
$c(A) = 2\times3$ matrices
} closed system (result always 2×3 matrices)
↓
it's vector space

Vector space

↳ a set of vector, where:
⓪ $u+v$ is in V (addition of u and v is in V)
① $u+v = v+u$
③ $u+(v+w) = (u+v)+w$
④ has zero vector
⑤ every u in V, there is $-u$ where $u-u=0$
⑥ cu is in V (scalar multiplication)
⑦ $c(u+v) = cu+v$
⑧ $(c+d)u = cu+du$
⑨ $c(du) = (cd)u$
⑩ $1(u) = u$

eg: set of all 2^{nd}-degree polynomials
↳ $p(x) = x^2$
$q(x) = 1+x-x^2$
} $p(x)+q(x) = 1+x$
sum is not 2nd degree polynomial
↓
not a vector space

* To prove it's vector space,
① The sum is in the set
② The scalar is in the set
} closed system

What is a vector space?

→ combination of elements that is:

① closed under scalar multiplication
→ $\vec{a} \in V$, c is scalar, then $c\vec{a} \in V$

② closed under scalar addition
→ $\vec{a} \in V$, $\vec{b} \in V$ then $\vec{a}+\vec{b} \in V$

③ contains zero vector
→ $0 \in V$

} How to effectively describe a vector space?

Basis

→ necessary vectors (the most minimum no. of vectors) that all vectors in the vector space can be formed by scalar addition of it

it must:

① linearly independant
→ each vectors in basis can't be produced by other vectors in the basis

② span the vector space
→ every vectors in V can be formed by scalar addition of it

eg: show that set of $\mathbb{R}$ (real number) is a vector space

① contain 0 vector? ✓
- 0 is real number,

② closed under vector addition? ✓
- let a and b are real numbers,
$a+b$, real num + real num is still real num

③ closed under vector multiplication? ✓
- let a is a real number
ca is still a real number

eg: Prove that $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ is basis for $\mathbb{R}^3$

① check for linearly independence
- Find out solution for homogenous system
~~r~~ref

$$\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right) \rightarrow \begin{array}{l}c_1=0\\c_2=0\\c_3=0\end{array}$$

} if only has trivial solution, it is linearly independent
→ no vector can be formed by other vector

→ the solution is also called Nullspace, the dimension is Nullity

* eg: $\left(\begin{array}{ccc|c}1&0&0&0\\0&1&1&0\\0&0&0&0\end{array}\right)$

→ $c_1=0$, $c_2+c_3=0$
$c_2=c_3$ (they can be represented by other vector = linearly dependent)

② Check for span

$Au=B$ - it must have soln. (consistent)

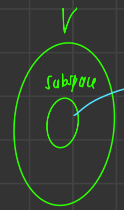
$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

since it's consistent, any vector in $\mathbb{R}^3$ can be formed by scalar addition of the basis

∴ it is basis for $\mathbb{R}^3$

* since it has 3 vectors as its basis, its dimension, $\dim(V)=3$

What is subspace?



it still have the same properties as vector space

What is span?

eg: $\mathbb{R}^3$, $v_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$ $v_3 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

$$\text{span}(v_1, v_2, v_3) = av_1 + bv_2 + cv_3$$

$$= \begin{pmatrix} 2a \\ a \\ -a \end{pmatrix} + \begin{pmatrix} 0 \\ 2b \\ 2b \end{pmatrix} + \begin{pmatrix} -c \\ -c \\ -c \end{pmatrix}$$

$$= \begin{pmatrix} 2a - c \\ a + 2b - c \\ -a + 2b - c \end{pmatrix}$$

span is any linear combination possible of the vectors

it is also the subspace of the vector

Row space and column space

$$A = \begin{pmatrix} \boxed{-3} & 6 & \boxed{-1} & 1 & 7 \\ \boxed{1} & -2 & \boxed{2} & 3 & -1 \\ \boxed{2} & -4 & \boxed{5} & 8 & -4 \end{pmatrix} \overset{\text{EROs}}{\sim} \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

reduced row-echelon form

rref(A)

Row space — all possible linear combinations of rows

non-zero

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Column space — all possible linear combinations of columns

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns (first non-zero element in each row)

let $c_1 = s, c_3 = t$

$s - 2c_2 - c_4 + 3c_5 = 0$

5-dimensional vectors

$\vec{u_1} = (1 \ -2 \ 0 \ -1 \ 3)$

$\vec{u_2} = (0 \ 0 \ 1 \ 2 \ -2)$

row space A

$\underline{\vec{u_1}, \vec{u_2}}$ form basis for row$(A) \subset \mathbb{R}^5$

$\vec{v_1} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$

$\underline{\vec{v_1}, \vec{v_2}}$ form basis for col$(A) \subset \mathbb{R}^3$

NO. of basis vectors = row rank of A = Column rank of A = rank of A = Basis of vector space of A = 2

Nullspace of a Matrix

↳ set of all solutions of the homogeneous system $Ax = 0$

↳ dimension of it is nullity

eg: find nullspace for $A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}$

$$A \xrightarrow{\text{ref}} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 + 2x_2 + 3x_4 = 0$$
$$x_3 + x_4 = 0$$

let $x_2 = s$, $x_4 = t$

$x_1 = -2s - 3t$

$x_3 = -t$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

basis of nullspace $A = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Change of basis

eg1: find coordinate of $u = (-2, 1, 3)$ relative to standard basis, $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

$$u = -2(1,0,0) + 1(0,1,0) + 3(0,0,1)$$

$$[u]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

eg2: Find the coordinate of $u = (1, 2, -1)$ relative to $B' = \{(1,0,1), (0,-1,2), (2,3,-5)\}$

$$c_1u_1 + c_2u_2 + c_3u_3 = (1, 2, -1)$$
$$c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5) = (1,2,-1)$$
$$(c_1 + 2c_3, -c_2 + 3c_3, c_1 + 2c_2 - 5c_3) = (1, 2, -1)$$

↓

$$c_1 + 2c_3 = 1$$
$$-c_2 + c_3 = 2$$
$$c_1 + 2c_2 - 5c_3 = -1$$

↓

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$u = 5(1,0,1) + (-8)(0,-1,2) + (-2)(2,3,-5)$$

↓

$$[u]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

Transition matrix

$[u]_{B'} = \boxed{P^{-1}}[u]_B$

transition matrix from $B \to B'$

eg: Find transition matrix, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to $B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$

$[B' \; B]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$ EROs

$[I_3 \; P^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}$$

Linear differential eqn

eg: for $y^{(4)} - y = 0$, check if e^x is a solution

$y = e^x$

$y^{(4)} = e^x$

$y^{(4)} - y = e^x - e^x = 0$

$\therefore y = e^x$ is solution for $y^{(4)} - y = 0$ #

Wronskian test for linear independence

$$W(y_1, y_2, \dots y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

— if { ① $\neq 0$, linearly independant
② $= 0$, linearly dependant

eg: Test solution set, $\{e^{-3x}, 3e^{-3x}\}$ for linear independance

$$W(e^{-3x}, 3e^{-3x}) = \begin{vmatrix} e^{-3x} & 3e^{-3x} \\ -3e^{-3x} & -9e^{-3x} \end{vmatrix}$$

$$= (e^{-3x} \cdot -9e^{-3x}) - (-3e^{-3x} \cdot 3e^{-3x})$$

$$= -9e^{-6x} + 9^{-6x}$$

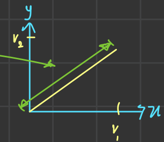
$$= 0$$

$\therefore$ since the Wronskian is equal to 0, e^{-3x} and $3e^{-3x}$ is not linearly independant #

Chapter 5: Inner Product Spaces

5.1 length and dot product

- Vector length, $\|V\| = \sqrt{V_1^2 + V_2^2}$



if $\|V\| = 1$, V is unit vector

distance of 2 vector: $d(v,u) = \|V\| - \|u\|$

- Unit vector, $u = \dfrac{V}{\|V\|}$

$[u_1 \ u_2] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

geometric formula

- Dot product, $u \bullet v = u_1 v_1 + u_2 v_2 = \|u\| \|V\| \cos\theta$

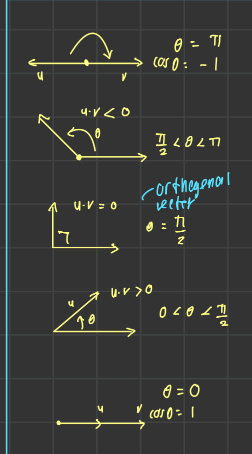
absolute value

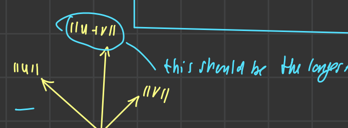
- Cauchy-Schwarz Inequality, $|u \cdot v| \leq \|u\| \|v\|$

A

- Angle between two vectors, $\cos\theta = \dfrac{u \cdot v}{\|u\| \|v\|}$, $0 \leq \theta \leq \pi$

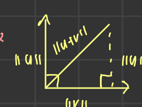
H





- Triangle inequality, $\|u + v\| \leq \|u\| + \|v\|$

- Pythagorean theorem, u and v are orthogonal if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$



5.2 Inner product spaces

the formula can be anything, depends on the properties of the inner product space

inner product, $\langle u, v \rangle$ must follow: ← general form for vector space V

① $\langle u,v \rangle = \langle v,u \rangle$

② $\langle u, v+w \rangle = \langle u,v \rangle + \langle u,w \rangle$

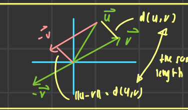
③ $c\langle u,v \rangle = \langle cu, v \rangle$

④ $\langle v,v \rangle \geq 0$, $\langle v,v \rangle = 0$ only if $v = 0$

↳ dot product, $u \cdot v$ is inner product for R^n — Euclean inner product

- length / distance of u, $\|u\| = \sqrt{\langle u,u \rangle}$

- distance between u and v, $d(u,v) = \|u - v\|$



angle between u and v, $\cos\theta = \dfrac{\langle u,v \rangle}{\|u\| \|v\|}$, $0 \leq \theta \leq \pi$

- u and v orthogonal, $\langle u,v \rangle = 0$

- $\|u\| = 1$, u is unit vector

- $u = \dfrac{v}{\|v\|}$ is unit vector in direction

- inner product of functions in $V([a,b])$, $\langle f,g \rangle = \int_a^b f(x) g(x) dx$

- eg: Prove Cauchy-Schwarz inequality for $f(x) = 1$, $g(x) = x$ in $V([0,1])$

① find $\langle f,g \rangle$

$\langle f,g \rangle = \int_0^1 x \, dx = \frac{1}{2}$

② find $\|f\| \|g\|$

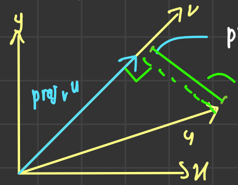
$\|f\|^2 = \langle f,f \rangle = \int_0^1 dx = 1$

$\|g\|^2 = \langle g,g \rangle = \int_0^1 x^2 dx = 1/3$

$\|f\| \|g\| = 1/\sqrt{3}$

$\therefore |\langle f,g \rangle| \leq \|f\| \|g\|$ shown

Orthogonal projections



orth. proj. of u onto v

$$\text{proj}_v u = av = \frac{\langle u, v \rangle}{\langle v, v \rangle} v = \frac{u \cdot v}{v \cdot v} v$$

$d(u, \text{proj}_v u) < d(u, cv)$

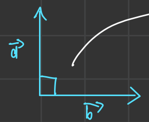
distance between orth. proj is the shortest

5-3 Ortonormal Basis: Gram-Shchmidt Process

Orthogonality

↳ $\vec{a}\cdot\vec{b} = 0$

$\theta = \frac{\pi}{2}$ rad



$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\underbrace{\cos\theta}_{\cos\frac{\pi}{2}=0} = [a_x \ \ a_y]\begin{bmatrix} b_x \\ b_y \end{bmatrix} = 0}$$

eg: $\vec{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$= 4(1) + 2(-3) - 1(-2)$

Orthonormality

$\frac{\vec{a}}{|\vec{a}|} = \hat{a}$ ← unit vector (vector with length 1)

↳ ① $|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = 1$ (vectors have length 1)

② orthogonal

eg: $\vec{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{16+4+1} = \sqrt{21}$

$|\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{1+9+4} = \sqrt{14}$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{21}}\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ -1/\sqrt{21} \end{bmatrix}$ ← length = 1

$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$ ← length = 1

normalization

Orthogonality of subspace

eg1: Square matrix, $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 0$ — orthogonal ✓

length $\sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2} = 1$ — orthonormal ✓

length $\sqrt{(1/\sqrt{2})^2 + (-1/\sqrt{2})^2} = 1$

if the square matrix, O is orthogonal, $O^{-1} = O^T$

$O^{-1} = O^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

eg2: functions, $f(x) = x$, $g(x) = 1$

$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx = \int_a^b x\,dx = \frac{x^2}{2}\Big|_a^b$

if $C[-1,1]$

$\frac{x^2}{2}\Big|_{-1}^{1} = \frac{(1)^2}{2} - \frac{(-1)^2}{2} = 0$

↳ orthogonal over -1 to 1

if $C[0,1]$

$\frac{x^2}{2}\Big|_0^1 = \frac{(1)^2}{2} - \frac{(0)^2}{2} = \frac{1}{2}$

↳ not orthogonal over 0 to 1

Gram-Schmidt process

original basis $\qquad\qquad$ orthogonal basis

$\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n \longrightarrow \vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_n$

R converting non-orthogonal basis into orthogonal basis

$$\boxed{\vec{u}_k = \vec{v}_k - \sum_{i=1}^{k-1} \frac{\langle \vec{v}_k, \vec{u}_i \rangle}{\|\vec{u}_i\|^2} \vec{u}_i}$$

1) $\vec{u}_1 = \vec{v}_1$ — u_1 has 1 term

2) $\vec{u}_2 = \vec{v}_2 - \dfrac{\langle \vec{v}_2, \vec{u}_1 \rangle}{|\vec{u}_1|^2} \vec{u}_1$ — u_2 has 2 terms

3) $\vec{u}_3 = \vec{v}_3 - \dfrac{\langle \vec{v}_3, \vec{u}_1 \rangle}{|\vec{u}_1|^2} \vec{u}_1 - \dfrac{\langle \vec{v}_3, \vec{u}_2 \rangle}{|\vec{u}_2|^2} \vec{u}_2$ — u_3 has 3 terms

we're subtracting any part of v that's not orthogonal to the new vectors to leave just the orthogonal part

eg: R^3: $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

1) $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

2) $\vec{u}_2 = v_2 - \dfrac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \dfrac{1(1) + 0(-1) + 1(1)}{1(1) - 1(-1) + 1(1)} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

3) $\vec{u}_3 = v_3 - \dfrac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 - \dfrac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \dfrac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \dfrac{5}{2} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

orthogonal basis = $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

orthonormal basis

$|\vec{u}_1| = \sqrt{\vec{u}_1 \cdot \vec{u}_1} = \sqrt{3}$

$|\vec{u}_2| = \sqrt{\vec{u}_2 \cdot \vec{u}_2} = \sqrt{2/3}$

$|\vec{u}_3| = \sqrt{\vec{u}_3 \cdot \vec{u}_3} = \sqrt{1/2}$

$\longrightarrow \overbrace{\dfrac{\vec{u}_1}{\sqrt{3}}, \dfrac{\vec{u}_2}{\sqrt{2/3}}, \dfrac{\vec{u}_3}{\sqrt{1/2}}}^{\text{convert to unit vector}} \longrightarrow \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$

5.4 Mathematical models and least square analysis

Orthogonal subspace

eg, $S_1 = \text{span}\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \right\}$ — subspace, $S_2 = \text{span}\left\{ \begin{bmatrix}-1\\1\\1\end{bmatrix} \right\}$ — subspace

orthogonal complement $S^\perp$ for S_1: v_1, v_2

u_1: orthogonal complement $S^\perp$ for S_1

$v_1 \cdot u_1 = 0$
$v_2 \cdot u_1 = 0$

$v_1 \cdot u_1 = 0$ for all v in S_1 and u in S_2

eg: Find the orthogonal complement for a subspace of R^4, $S = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

not leading one, choose as free variables

any value can be chosen, but in order to standardize the param. representation

$S^T S^\perp = 0$

Find the null space (homogenous system)

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = 0$$

let $u_2 = s$, $u_3 = t$

$u_1 = -2s - t$

$u_4 = 0$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} -2s-t \\ s \\ t \\ 0 \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$S^\perp = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ S and $S^\perp$ form a basis for R^4

→ $R^n = A \oplus U$

→ R^n is the direct sum of A and U

Projection into subspace

if $\{u_1, u_2, \ldots, u_t\}$ is orthonormal basis for subspace of R^n, $v \in R^n$

- $\text{proj}_s v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \ldots + (v \cdot u_t)u_t$
- $S^T V u = S^T S$

use finding least square regression method

Fundamental subspace of a matrix

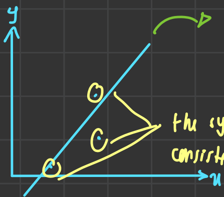
$N(A)$ = nullspace of A	$N(A^T)$ = nullspace of A^T
$R(A)$ = column space of A	$R(A^T)$ = column space of A^T

if A is $m \times n$ matrix,

① $R(A)$, $N(A^T)$ = orth. subspace of R^m → $R^m = R(A) \oplus N(A^T)$

② $R(A^T)$, $N(A)$ = orth. subspace of R^n → $R^n = R(A^T) \oplus N(A)$

Least square regression line



→ least square regression line

↳ u values that minimizes the error of $Au = b$

the system aren't consistent

$$\boxed{Au = B \rightarrow A^TAu = A^TB}$$

eg: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$A^TB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

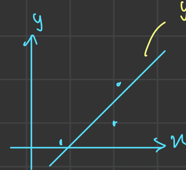
$A^TAu = A^TB$

$\Downarrow$

$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

$u = \begin{bmatrix} -5/3 \\ 3/2 \end{bmatrix}$

$y = -\frac{5}{3} + \frac{3}{2}u$



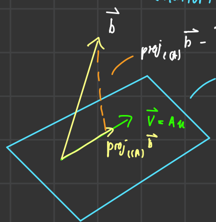
Least square approximation

- No solution for $A\vec{x} = \vec{b}$, $\vec{b}$ is not in the $C(A)$ (column space of A)
- We want to find $\boxed{\vec{x}}$ where $\|\vec{b} - \boxed{A\vec{x}}\|$ is as minimal (to approximate)

the least square solution

difference is as little

$$A\vec{x} = proj_{C(A)}\vec{b}$$



Orthogonal to $C(A)$; orthogonal complement

$$\boxed{A\vec{x} - \vec{b}} = \boxed{proj_{C(A)}\vec{b} - \vec{b}}$$

these are $\in C(A)^{\perp}$ — orthogonal complement $(C(A) \cdot C(A)^{\perp} = 0)$

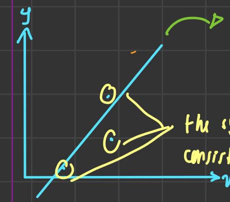
$C(A)^{\perp} = N(A^T)$, therefore $A\vec{x} - \vec{b} \in N(A^T)$

$$A^T(A\vec{x} - \vec{b}) = 0$$

$$\boxed{A^T \boxed{A\boxed{\vec{x}}} = A^T\vec{b}}$$

the least square approximation

$proj_{C(A)}\vec{b}$



least square regression line

↳ x values that minimizes the error of $Ax = b$

the system aren't consistent

$$\boxed{Ax = B \rightarrow A^TAx = A^TB}$$

eg: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$

$A^TB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$

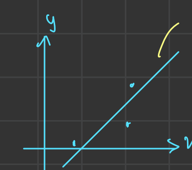
$$A^TAx = A^TB$$

$$\Downarrow$$

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$x = \begin{bmatrix} -5/3 \\ 3/2 \end{bmatrix}$$

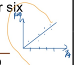
$$y = -\frac{5}{3} + \frac{3}{2}x$$



Application of least square approximation - mathematical modeling

eg:

The table shows the world population (in billions) for six different years. (Source: U.S. Census Bureau)



Year	1985	1990	1995	2000	2005	2010
Population, y	4.9	5.3	5.7	6.1	6.5	6.9

(since population growth isn't perfectly linear, find the linear line that closely estimate the growth)

Let $x = 5$ represent the year 1985. Find the least squares regression quadratic polynomial $y = c_0 + c_1x + c_2x^2$ for the data and use the model to estimate the population for the year 2020.

$$c_0 + 5c_1 + 25c_2 = 4.9$$

$$c_0 + 10c_1 + 100c_2 = 5.3$$

$$c_0 + 15c_1 + 225c_2 = 5.7$$

$$c_0 + 20c_1 + 400c_2 = 6.1$$

$$c_0 + 25c_1 + 625c_2 = 6.5$$

$$c_0 + 30c_1 + 900c_2 = 6.9$$

$$\Downarrow \; Ax = b$$

$$\begin{bmatrix} 1 & 5 & 25 \\ 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \\ 1 & 25 & 625 \\ 1 & 30 & 900 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4.9 \\ 5.3 \\ 5.7 \\ 6.1 \\ 6.5 \\ 6.9 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 6 & 105 & 2275 \\ 105 & 2275 & 55{,}125 \\ 2275 & 55{,}125 & 1{,}421{,}875 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 35.4 \\ 654.5 \\ 14{,}647.5 \end{bmatrix}$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 0.08 \\ 0 \end{bmatrix}$$

$\therefore$ at $x = 40$, $y = 4.5 + 0.08(40) = 7.7$ billion

Chapter 5: Eigenvalues and Eigenvectors

$$A\vec{u} = \lambda \vec{u}$$

eigenvector

eigenvalue

→ when matrix A is multiplied with $\vec{u}$, it produce $\lambda \vec{u}$ where λ is scalar

eg: $A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A\vec{u} = \begin{bmatrix} -3+1 \\ -2+0 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

λ (eigenvalue)

$\vec{u}$ (eigenvector)

Finding eigenvalues, eigenvectors

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}$$

① find value of λ by using formula $\boxed{|A - \lambda I| = 0}$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - (4)(1) = 0$$

$$1 - 2\lambda + \lambda^2 - 4 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

eigenvalues : $\lambda = 3$, $\lambda = -1$

② finding eigenvectors using $\boxed{(A - \lambda I)\vec{u} = 0}$

for $\lambda = 3$,

$$\begin{bmatrix} 1-(3) & 1 \\ 4 & 1-(3) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

↓ eros

$$\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

for $\lambda = -1$

$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

↓ eros

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

For triangular matrices, the main diagonal is its eigenvalues

$$\begin{bmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_1 & 0 & 0 \\ * & \lambda_2 & 0 \\ * & * & \lambda_3 \end{bmatrix}$$

eigenvalues : $\lambda_1, \lambda_2, \lambda_3$

$$\text{eigenspace} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

logical explanation for $\boxed{|A - \lambda I| = 0}$

$A\vec{x} = \lambda\vec{x}$

$\downarrow$

$A\vec{x} - \lambda I\vec{x} = \vec{0}$

identity, won't change the value

$\downarrow$ we want non-trivial soln, $(\vec{x} \neq \vec{0})$

$(A - \lambda I)\vec{x} = \vec{0}$

if $(A - \lambda I)$ is invertible

$\vec{x} = (A - \lambda I)^{-1}\vec{0}$

$\vec{x} = 0$

therefore, $(A - \lambda I)$ can't be invertible

non-invertible, $\det(A) = 0$

$|A - \lambda I| = 0$

Application of eigenvalues and eigenvectors

Diagonalization

diagonal matrix $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

$$A = \underline{U D U^{-1}}$$

writing a matrix as a product of matrices

Finding D

A can be obtained from D from change of basis (P) = A is similar to D

$\boxed{\text{the positions are important!}}$

$$U^{-1} A U = D$$

made of eigenvalues of A $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

- matrix U diagonalizes A
- made of eigenvectors of A

made of eigenvectors of A

$$\begin{array}{ccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{array}$$
$$\begin{pmatrix} u_1 & u_1 & u_1 \\ u_2 & u_2 & u_2 \\ u_3 & u_3 & u_3 \end{pmatrix}$$

* $n \times n$ matrix is diagonalizable if it has $\boxed{n \text{ eigenvectors}}$

how to know?

① has n unique eigenvalues

or

② has repeated eigenvalues, but total of n linearly independant eigenvectors

We need $n \times n$ matrix to represent $n \times n$ matrix

3 distinct eigenvalues

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

eg: $A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$, represent $A = UDU^{-1}$

① Find the eigenvalues, by using $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & -4 \\ 5 & 6-\lambda \end{vmatrix} = 0$$

$(\lambda - 1)(\lambda - 2) = 0$

$\lambda = 1, \lambda = 2$

② Find eigenvectors

for $\lambda = 1$,

$$(A - (1)I)\vec{u}_1 = \begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\vec{u}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

for $\lambda = 2$,

$$(A - (2)I)\vec{u}_2 = \begin{bmatrix} -5 & -4 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\vec{u}_2 = \begin{bmatrix} -4/5 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad U = \begin{bmatrix} -1 & -4/5 \\ 1 & 1 \end{bmatrix} \quad U^{-1} = \begin{bmatrix} -5 & -4 \\ 5 & -5 \end{bmatrix}$$

λ_1, λ_2; $\vec{u}_1$, $\vec{u}_2$

$$A = UDU^{-1} = \begin{bmatrix} -1 & -4/5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -5 & -4 \\ 5 & -5 \end{bmatrix}$$

Symmetric matrices

$A = A^T$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

eigenvectors, for symmetric matrices

$\bar{u}_1 = \begin{bmatrix} s \\ -s \end{bmatrix}$, $\bar{u}_2 = \begin{bmatrix} t \\ t \end{bmatrix}$

are orthogonal, $\bar{u}_1 \cdot \bar{u}_2 = 0$

Orthogonal diagonalization

A is orthogonally diagonalizable if A is orthogonal matrix $\rightarrow A A^T = I_n$

$A^T = A^{-1}$

eg: Find P that orthogonally diagonalizes $A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$

① Find eigenvalue

$$|\lambda I - A| = \begin{vmatrix} \lambda+2 & -2 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda+3)(\lambda-2)$$

② Find eigenvector, convert it into orthonormal

since A is symmetric, the eigenvectors are orthogonal

$\lambda = -3$, $\bar{u}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, orthonormal $\bar{u}_1 = \dfrac{1}{\|(-2,1)\|}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{-2^2+1^2}}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\lambda = 2$, $\bar{u}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, orthonormal $\bar{u}_2 = \dfrac{1}{\|(1,2)\|}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \dfrac{1}{\sqrt{2^2+1^2}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

③ For each eigenvalue of multiplicity $k \geq 2$, find set of k linearly independant using Gram-Schmidt orthonormalization (there's no such case here, so skip)

④ using p1, p2 as column vectors construct P

$$D = P^{-1}AP = P^TAP$$

$$= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

Chapter 6: Linear Transformation

↳ $\vec{v} \to L(v) \to \vec{w}$

$$V \xrightarrow{L} W$$

$L: V \to W$

↳ map one vector space onto another

- may also map to the same vector space

A linear transformation must:

① $L(c\vec{v}) = cL(\vec{v})$

② $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$

eg: $L: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad L(\vec{v}) = \begin{bmatrix} v_2 \\ v_1+v_2 \\ v_1-v_2 \end{bmatrix}$

① $L(c\vec{v}) = cL(\vec{v})$? ✓

$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$

$$L(c\vec{v}) = \begin{bmatrix} cv_2 \\ cv_1+cv_2 \\ cv_1-cv_2 \end{bmatrix} = \begin{bmatrix} c(v_2) \\ c(v_1+v_2) \\ c(v_1-v_2) \end{bmatrix} = c\begin{bmatrix} v_2 \\ v_1+v_2 \\ v_1-v_2 \end{bmatrix} = cL(\vec{v})$$

② $L(\vec{v}+\vec{w}) = L(\vec{v}) + L(\vec{w})$? ✓

$V+W = \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix}$

$$L(\vec{v}+\vec{w}) = \begin{bmatrix} v_2+w_2 \\ (v_1+v_2)+(w_1+w_2) \\ (v_1-v_2)+(w_1-w_2) \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1+v_2 \\ v_1-v_2 \end{bmatrix} + \begin{bmatrix} w_2 \\ w_1+w_2 \\ w_1-w_2 \end{bmatrix} = L(\vec{v}) + L(\vec{w})$$

$L(\vec{v}) = \begin{bmatrix} v_2 \\ v_1+v_2 \\ v_1-v_2 \end{bmatrix}$ is a linear transformation

Representing linear transformation as matrices, $L(\vec{v}) = A\vec{v}$

$L: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad L(\vec{v}) = \begin{bmatrix} v_2 \\ v_1+v_2 \\ v_1-v_2 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ std. basis

$L\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$L\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$A\vec{v} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1+v_2 \\ v_1-v_2 \end{bmatrix}$$

$V \xrightarrow{L} W$

image

↳ the result of linear transformation of $S = L(S)$

- image of entire vector = range of L

eg: $L: \mathbb{R}^3 \to \mathbb{R}^2 \quad L(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix}$

subspace of $V = \begin{bmatrix} c \\ 2c \\ 0 \end{bmatrix}$

$$L\begin{bmatrix} c \\ 2c \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix}$$

image of subspace S

kernel of L ; ker(L)

↳ set of vectors in V that gives the 0 vector in W, $L(V) = 0$

eg: $L: \mathbb{R}^3 \to \mathbb{R}^2 \quad L(\vec{v}) = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix}$

$$L\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 - v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

ker(L)

any vector where $v_1 = 0$, and $v_2 = v_3$ will result in zero vec