Week 1 : Sets

Set

↳ a collection of object that has the same value / characteristics

↓ notation

$A = \{1, 2, 3\}$

$\mathbb{R} = \{x, x \in \mathbb{R}\}$ - set of real number

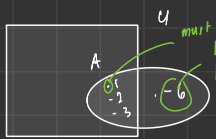
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ - infinite set

Venn Diagrams

↳ provides pictorial views of set

U = set of +ve integers

$A = \{1, 2, 3, -6\}$



must include ∴ not a +ve, put outside of U

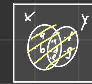
represent number of element inside

Operation of sets

① Union, $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$

$X = \{a, b, c, d, e\}$, $Y = \{c, d, e, f, g\}$

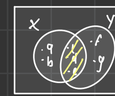
$X \cup Y = \{a, b, c, d, e, f, g\}$



② Intersection, $X \cap Y = \{x \mid x \in X \text{ and } x \mid x \in Y\}$

$X = \{a, b, c, d, e\}$, $Y = \{c, d, e, f, g\}$

$X \cap Y = \{c, d, e\}$

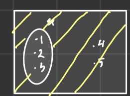


③ Complement, $\bar{x}$ or $x^c = \{x \mid x \in U \text{ and } x \notin X\}$

$U = \{1, 2, 3, 4, 5\}$

$x = \{1, 2, 3\}$

$\bar{x} = \{4, 5\}$



④ Difference, $X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$

↳ remove elements from X that ∈ Y

$X = \{1, 3, 5\}$

$Y = \{4, 5, 6\}$

$X - Y = \{1, 3\}$

$X - Y \neq Y - X$

Theorem

let U be a universal set, let $A, B, C \subseteq U$

1) Associate law

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

2) Commutative Law

$A \cup B = B \cup A$

$A \cap B = B \cap A$

3) Distributive Law

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4) Identity Law

$A \cup \emptyset = A$, $A \cap U = A$

5) Idempotent Law

$A \cup \bar{A} = U$, $A \cap \bar{A} = \emptyset$

6) Bound Law

$A \cup U = U$, $A \cap \emptyset = \emptyset$

7) De Morgan's Law

$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$, $\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

7, 8, 9

$A = \{1, 2, 3\}$ 789 =

$B = \{4, 5, 6\}$

Week 2 : Set Theory

if

① finite, not too large

list all the elements,

$A = \{1, 2, 3, 4\}$

② infinite, or large finite

list a property necessary

$A = \{x \mid x \text{ is Rational Number}\}$

Properties of set

① Empty set

$A = \varnothing$ or $A = \{\}$

② Equal set

$X = Y$, $x \in X$, $x \in Y$

and

$x \in Y$, $x \in X$

eg: $A = \{1, 3, 2\}$, $B = \{2, 3, 2, 1\}$, show $A = B$

for 1, $1 \in A$, $1 \in B$ and for 2, $2 \in B$, $2 \in A$

2, $2 \in A$, $2 \in B$ 3, $3 \in B$, $3 \in A$

3, $3 \in A$, $3 \in B$ 2, $2 \in B$, $2 \in A$

1, $1 \in B$, $1 \in A$

$\therefore A = B$

* if $x \notin A$, no need to continue

③ Subset

↳ suppose that X and Y are sets, if every elements of X is an element of Y, we say that X is a subset of Y, $X \subseteq Y$

eg 1

$X = \{1, 3\}$, $Y = \{1, 2, 3, 4\}$

since $1 \in X$, $1 \in Y$

$3 \in X$, $3 \in Y$

$\therefore$ Thus $X \subseteq Y$

eg 2

find subset of $X = \{1, 2\}$

$\varnothing, \{1\}, \{2\}, \{1, 2\}$

no. of elements : $2^n = 2^2 = 4$

EG

$X = \{1, 2, 3\}$, for any subset of X, say Y, $Y \subseteq X$

a) $Y \cup X = X$

b) $Y \cap X = Y$

c) $Y - X = \varnothing$

$Y = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

a) $Y \cup X$,

$\varnothing \cup \{1, 2, 3\} = \{1, 2, 3\} = X$

$\{1\} \cup \{1, 2, 3\} = \{1, 2, 3\} = X$

$\vdots$

$\{1, 2, 3\} \cup \{1, 2, 3\} = \{1, 2, 3\} = X$

b) $Y \cap X$

$\varnothing \cap \{1, 2, 3\} = \varnothing = Y$

$\{1\} \cap \{1, 2, 3\} = 1 = Y$

$\vdots$

$\{1, 2, 3\} \cap \{1, 2, 3\} = X = Y$

c) $Y - X$

$\varnothing - \{1, 2, 3\} = \varnothing$

$\{1\} - \{1, 2, 3\} = \varnothing$

$\vdots$

$\{1, 2, 3\} - \{1, 2, 3\} = \varnothing$

Week 3: Set Logic

Statement / Proposition

↳ Sentence that has a truth value

True / False

Connective

↳ Symbol or word to combine two or more statements into a compound proposition

① Conjunction (and) : $\wedge$

② Disjunction (or) : $\vee$

③ Negation (not) : $\neg$

④ Implication (if... then): $\longrightarrow$

⑤ By conditional (if and only if): $\longleftrightarrow$

① Conjunction / and / $\wedge$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

∴ P and Q is true if both are true

② Disjunction / or / $\vee$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

∴ $P \vee Q$ is true if one is true

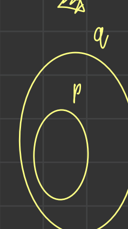
④ Implication, If... then, $\rightarrow$

- simple sentence : P, Q
- compound sentence : if P then Q
- symbolic form : $P \rightarrow Q$

P: hypothesis, Q: conclusion

P	Q	$P \rightarrow Q$	
T	T	T	→ rain, road wet
T	F	F	→ rain, road not wet
F	T	T	→ not rain, road wet
F	F	T	→ not rain, road not wet

- $P \rightarrow Q$
- Q if P
- P only if Q
- P is sufficient condition for Q
- Q is necessary condition for P



p is subset, all true for p must true for q

- p "1" q "0" is impossible = F
- p "0" q "1" is possible = T

maybe it's outside of p but still inside of q

③ Negation / not / $\neg$

↳ create new sentence from a sentence by prefixing the old sentence with not

↳ original sentence : p

negation : $\neg p$

eg: I like nasi lemak and 4 fingers

↳ $P \wedge Q$

negation: I hate nasi lemak or 4 fingers

$\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$

rule of negation: The connector of the sentence also changes

p	$\neg p$	$\neg\neg p$	$\neg\neg\neg p$
T	F	T	F
F	T	F	T

∴ $p \equiv \neg\neg p$

$\neg p \equiv \neg\neg\neg p$

Week 4 : Set & Logic

Tautology

↳ sentence that is always true

eg:

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

} Always true, = tautology

p	q	$p \wedge q$	$p \wedge q \to p$	
T	T	T	T	→ p is indeed true when $p \wedge q$
T	F	F	T	→ The statement is still true
F	T	F	T	→ 〃
F	F	F	T	→ 〃

Contradiction

↳ sentence that is always false

eg:

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

} always false = contradiction

Quantifiers

① Universal, $\forall$

↳ All, for all, for every, for each, anyone, everyone...

eg: All humans are mammals

$\forall x\,(Hx \to Mx)$ Hx : x is human
Mx : x is mammal

② Existential, $\exists$

↳ there is, there are, there exist, for some...

eg: There are things that grow

$\exists x\,(Tx \wedge Gx)$ Tx : x is a thing
Gx : x grow

Sentence

① True sentence: $1+1=2$

② False sentence : $1+1=0$

③ Open sentence : $x^2+y^2=1$

↳ can be true / false, depends on the replacement of x and y

Solution set

eg: $p(x)$: $x-1 \geq 0$

U : $\{0, 1, 2\}$

find the solution sets

x	$x-1$	$x-1 \geq 0$
0	-1	F
1	0	T
2	1	T

$\therefore SS(P(x)) = \{1, 2\} \neq U$

Truth value analysis

① Sentence with 1 variable

let $U = \{1/2, 1, 2, 3, 4\}$

Px : $x^2 \geq x$

determine the truth value of $\forall x\, Px,\ x \in U$

x	x^2	$x^2 \geq x$
½	¼	F
1	1	T
2	4	T
3	9	T
4	16	T

$SS(P(x)) = \{1, 2, 3, 4\} \neq U$

$\therefore \forall x\, Px$ is false

(for all x, $P(x)$ is false

② Sentence with 2 variables

let $U = \{1, 2, 3\}$

$P(x)(y) = x - y \leq 2$

determine the truth value of

$\forall x \forall y\ (x-y \leq 2)$, $x, y \in U$

x	y	$x-y$	$x-y \leq 2$
1	1	0	T
	2	-1	T
	3	-2	T

x	y	$x-y$	$x-y \leq 2$
2	1	1	T
	2	0	T
	3	-1	T

x	y	$x-y$	$x-y \leq 2$
3	1	2	T
	2	1	T
	3	0	T

$\therefore \forall x \forall y\, (Pxy)$ is true

③ Sentence with infinite set

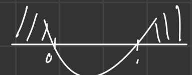
let $U = \mathbb{R}$, determine the truth of

$\forall x\,(x^2 \geq x)$

$SS(x^2 \geq x)$

$SS(x^2 - x \geq 0)$

$x(x-1) = 0$, $x = 0$, $x = 1$



$x \leq 0$ U $x \geq 1$

$\therefore SS(x^2 \geq x) = \{x \in \mathbb{R} ; x \geq 1 \text{ or } x \leq 0\}$

$SS(x^2 \geq x) \neq U$

Week 5: Counting Principle

Multiplication principle

$\underline{n_1 \text{ choice}}$ $\underline{n_2 \text{ choice}}$

possible choice $= n_1 \times n_2$

Permutation, ${}^nP_r$

subsidiary

- ordering of object
- $\boxed{\text{if } n = r, \text{ permutation} = n(n-1)(n-2)\ldots(1) = n!}$
- eg: How many permutations of the letter ABCDEF contain the substring "DEF"

$\boxed{\boxed{DEF}\ \underline{3}\ \underline{2}\ \underline{1}} = \; = \boxed{{}^1P_1} \times \boxed{{}^4P_4} = 1! \cdot 4! = 24$

$u = \{(DEF), A, B, C\}$

or

$\boxed{DEF}\ \boxed{\underline{3}\ \underline{2}\ \underline{1}} = 6$
+
$\underline{3}\ DEF\ \underline{2}\ \underline{1} = 6$
+
$\underline{3}\ \underline{2}\ DEF\ \underline{1} = 6$
+
$\underline{3}\ \underline{2}\ \underline{1}\ DEF = 6$

$= \boxed{{}^1P_1} \times \boxed{{}^3P_3} \times \boxed{4} = 1! \cdot 3! \cdot 4 = 24$

r-Permutation / sub-permutation, $P(n,r)$

- not all elements are chosen, $r \neq n$

$$\boxed{\begin{aligned} P(n,r) &= n(n-1)(n-2)\ldots(n-r+1) \\ &= \frac{n!}{(n-r)!} \rightarrow r \leq n \end{aligned}}$$

- eg: in how many ways can we select a chairperson, vice-chairperson, secretary, and treasurer from a group of 10 persons?

$p(10,4) = {}^{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$

or

$p(10,4) = {}^{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$

- eg2: Al, Bob, Carl, Dan, Ed, Frank. How many ways can they sit if Al, Bob, Carl shouldn't sit together?

= Total of ways − Al, Bob, Carl sit together

$= {}^6P_6 - {}^3P_3 \cdot {}^4P_4$

$= 576$

Combination, $C(n,r)$ / $\binom{n}{r}$ / ${}^nC_r$

- Selecting from a set (order is not important)
- $\boxed{{}^nC_r = \frac{n!}{(n-r)! \cdot r!}}$
- eg: A group of five, M, B, R, A, N has decided to form a taskforce of 3.

$\binom{5}{3} = {}^5C_3 = \frac{5!}{(5-3)! \cdot 3!} = 10$

- eg3: In how many ways can letter $\boxed{SUCCESS}$ be arranged if no two s's are next to each other?

There are 3S, 2C = identical

_ U _ E _ C_1 _ C_2 _

$= \frac{4!}{2!} \cdot \frac{5 \cdot 4 \cdot 3}{3!}$

$= 120$

Week 6 : Generalized Permutation and Combination

Permutation with identical objects

- Suppose that there is a sequence S of 'n' items having n_1 identical object of type 1, n_2 identical object of type 2.

The number of ordering of S is:

$$\boxed{\frac{n!}{n_1! \cdot n_2!}}$$

eg: how many strings can be formed from "MISSISSIPPI"?

letters: M-1, I-4, S-4, P-2,

for M, ${}^{11}C_1$
for I, ${}^{10}C_4$
for S, ${}^{6}C_4$
for P, ${}^{2}C_2$

$$= \frac{11!}{1!\,\cancel{10!}} \cdot \frac{\cancel{10!}}{4!\,\cancel{6!}} \cdot \frac{\cancel{6!}}{4!\,\cancel{2!}} \cdot \frac{\cancel{2!}}{2!}$$

$$= \boxed{\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!}} = 34650 \text{ possibilities}$$

T-elements, k- is selected, repetition is allowed

$$\boxed{C(k+t-1,\, k)}$$

From Pascal's triangle

eg: There is piles of red, blue, green balls, each piles contain at least 8 balls.

a) How many ways can we select 8 balls?

$k=8$, $t=3$

$C(8+(3-1), 3) = C(10, 3) = 45$ ways

b) How many ways to select 8 balls, we must at least select one ball for each color.

$\underbrace{\underline{R}\ \ \underline{G}\ \ \underline{B}}_{3 \text{ is chosen}}\quad \underbrace{\_\ \_\ \_\ \_\ \_}_{5 \text{ slots left}}$

$k=5$, $t=3$

$C(5+(3-1), 3) = C(7, 2) = 21$ ways

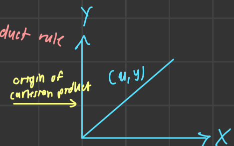
Week 8, 9: Relations

What is relation?

↳ a table that lists which elements of the first set relate to which elements of the second set

Cartesian product × AKA product rule

↳ if X and Y are sets, X×Y denote the set of all ordered pairs, (x,y) where $x \in X$, $y \in Y$



* $X \times Y \neq Y \times X$

eg: $X = \{1,2,3\}$, $Y = \{a,b\}$

$X \times Y = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

Binary Relation, R

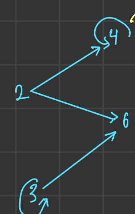
↳ binary relation R from a set X to a set Y is a subset of cartesian product X×Y. If $(x,y) \in R$, xRy

eg: let $X = \{2,3,4\}$, $Y = \{3,4,5,6,7\}$

$(x,y) \in R$: if y divides by x with no remainders

$R = \{(2,4),(2,6),(3,3),(3,6),(4,4)\}$ →

Diagraph of Relation

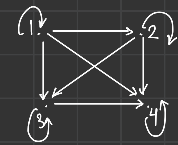


Properties of Relationship

① Reflexive – reflects itself

$(x,x) \in R$, for every $x \in X$

eg:



=

	1	2	3	4
1	1	1	1	1
2	0	1	1	1
3	0	0	1	1
4	0	0	0	1

To show R is reflexive,

$(1,1) \in R$ $(3,3) \in R$

$(2,2) \in R$ $(4,4) \in R$

② Anti-symmetric

if $(x,y) \in R$, $(y,x) \in R \rightarrow x = y$

④ Inverse

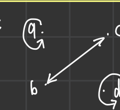
if $R = \{(x,y) \mid x,y \in R\}$ then $R^{-1} = \{(y,x) \mid y,x \in R\}$

if $R = \frac{x}{y}$, $R^{-1} = y \times x$

③ Symmetric , if... then...

$\forall x \forall y\ ((x,y) \in R \rightarrow (y,x) \in R)$

eg:



To show R is symmetric,

① $(a,a) \in R$
$(a,a) \in R$

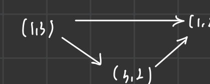
② $(c,b) \in R$
$(b,c) \in R$

③ $(c,b) \in R$
$(b,c) \in R$

④ $(d,d) \in R$
$(d,d) \in R$

④ Transitive

if $(x,y) \in R$, $(y,z) \in R \rightarrow (x,z) \in R$



Week 10: Matrices of Relations

How to write?

$R = \{(2,6), (2,8), (3,6), (4,8)\}$

if exist in R

$$\begin{array}{c|c} & y \rightarrow \\ x \downarrow & \begin{array}{cc} & \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \end{array}$$

Properties for matrix of relations

① Reflexive

↳ has 1's on the main diagonal

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

② Symmetric

↳ $\forall\ i,j \in R,\ j,i \in R$

(for all)

$a_{ij} = a_{ji} = 1$

$a_{ij} = a_{ji} = 0$

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

③ Transitive

↳ $A^2 = A$

after replacing non-zeros with 1

$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ A^2 \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

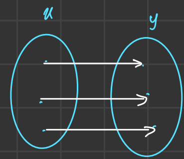
Week 12: Function

* Function is a relation between two sets that associates each element of first set (domain) to a unique element in the second set (codomain)

Properties of function

① One-to-one (1-1)

↳ there is only one x in the domain of $f(x)$ such that $f(x) = y$



eg: find out if $f(x) = 2x + 1$ is (1-1)

$$\therefore f(x_1) = f(x_2)$$

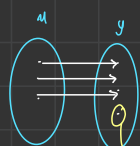
$$2x_1 + 1 = 2x_2 + 1$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$ → the function is (1-1)

② Onto

↳ for every $y \in Y$, there exist $x \in X$ such that $f(x) = y$



no x value map to it → not onto

eg: $f(x) = \frac{1}{x^2}$

$$y = \frac{1}{x^2}$$

$$x = \frac{1}{\pm\sqrt{y}}$$

$y > 0$

↑ if y isn't > 0, then there is no x value that maps to it → not onto

Bijection

↑ if it satisfies both properties

Week 14: Mathematical Induction

→ a technique to prove certain types of mathematical statements: general propositions which assert that something is true for all positive integers for all positive integers from some point on.

How to make a mathematical induction?

eg1: $S_n = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

by mathematical induction,

① P(1) is true : $1 = \frac{1(1+1)}{2} = 1$ ✓

② P(k) is true: $1+2+3+\ldots+k = \frac{k(k+1)}{2}$ ✓

③ P(k+1) is true: $\underbrace{1+2+3+\ldots+k}+(k+1) = \frac{(k+1)((k+1)+1)}{2}$

$\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$

$k(k+1)+2(k+1) = (k+1)(k+2)$

$k^2+k+2k+2 = k^2+2k+k+2$

$k^2+3k+2 = k^2+3k+2$

LHS = RHS ✓

eg 2: 6^n-1 is divisible by 5 by mathematical induction

① P(1) is true: $P(1) = \frac{6^1-1}{5} = \frac{5}{5} = 1$ ✓

② P(n) is true : $\boxed{6^n-1 \text{ is divisible by } 5}$

③ P(n+1) is true: $p(n+1) = 6^{n+1}-1$

$= 6^n \cdot 6 - 1$

$= 6(6^n) - (6-5)$

$= 6(6^n-1) + 5$ divisible by 5

divisible by 5

*eg: 5 is divisible by 5, 5n'll always be divisible by 5 for all n

all divisible by 5

eg 3: $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

by mathematical induction,

① P(1) is true: $P(1) = 1^2 = \frac{1^2(1^2+1)(2\cdot1^2+1)}{6} = 1$ ✓

② P(k) is true: $p(k) = \frac{k(k+1)(2k+1)}{6} = 1^2+2^2+3^2+\ldots+k^2$ ✓

③ P(k+1) is true: $\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)\cdot((k+1)+1)\cdot(2(k+1)+1)}{6}$

$k(k+1)(2k+1) + 6(k+1)^2 = (k+1)\cdot((k+1)+1)\cdot(2(k+1)+1)$

$k(2k^2+3k+1) + 6(k^2+2k+1) = (k^2+3k+2)(2k+3)$

$2k^3+3k^2+k+6k^2+12k+6$ = $2k^3+3k^2+6k^2+9k+4k+6$

$2k^3+9k^2+13k+6 = 2k^3+9k^2+13k+6$

LHS = RHS ✓