Topic 1 : Integration

Integration of exponential function

$\int e^x dx = e^x + C$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$

$\int \frac{1}{ax+b} dx = \frac{\ln(ax+b)}{a} + C$

Trigonometric Integration

$\int \cos ax\, dx = \frac{\sin ax}{a} + c$

$\int \sec ax \tan ax\, dx = \frac{1}{a} \sec ax + c$

$\int \sin ax\, dx = \frac{-\cos ax}{a} + C$

$\int \operatorname{cosec}^2 ax\, du = -\frac{1}{a} \operatorname{cotan} + C$

$\int \sec^2 ax\, dx = \frac{\tan ax}{c} + C$

$\int \operatorname{cosec} ax \operatorname{cotan} dx = -\frac{1}{a} \operatorname{cosec} ax + c$

$\int \sin^2 ax\, dx = \frac{1}{2} \int (1 - \cos 2ax) dx$

$\int \cos^2 ax\, dx = \frac{1}{2} \int (1 + \cos 2ax) dx$

Techniques of Integration

Substitution

let $u = 2x+1$

$du = 2dx$

$dx = \frac{du}{2}$

$$\int (2x+1)^5 dx = \int u^5 \frac{du}{2}$$
$$= \frac{1}{2} \int u^5 du$$
$$= \frac{1}{2} \cdot \frac{u^6}{6} + C$$
$$= \frac{(2x+1)^6}{12} + C$$

By Parts

$\int u\,dv = uv - \int v\,du$

L } - logarithmic
O
P - polynomial
E - exponential
T - trigonometry

$u = \ln(4x-3)$

$du = \frac{4}{(4x-3)} dx$

$dv = 2x\,dx$

$v = x^2$

$$\int 2x \ln(4x-3) dx = x^2 \ln(4x-3) - \int x^2 \cdot \frac{4}{(4x-3)} dx$$

d(x)

Q(x)

$$\begin{array}{r} x + 3/4 \\ 4x-3 \,\overline{)\, 4x^2 \qquad\quad} \\ -\ 4x^2 - 3x \quad \\ \hline 0 \quad 3x \qquad \\ -\qquad 3x - 9/4 \\ \hline 0 + 9/4 \end{array}$$

R(x)

$$= x^2 \ln(4x-3) - \int x + \frac{3}{4} + \frac{9/4}{4x-3} dx$$

$$= x^2 \ln(4x-3) - \frac{x^2}{2} - \frac{3}{4}x + \frac{9}{16} \ln|4x-3| + C$$

Partial Fraction

Proper fraction (power of f(x) < g(x))

$$\frac{x}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

$$\frac{1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$\frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$\frac{2}{(x^2+1)^2} = \frac{Ax B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Improper fraction (power of f(x) ≥ g(x))

$$\int \frac{x^2+3x-10}{x^2-2x-3} dx = 1 + \frac{5x-7}{x^2-2x-3}$$

$$\begin{array}{r} 1 \qquad\qquad \\ x^2-2x-3 \,\overline{)\, x^2+3x-10} \\ -\ \underline{x^2-2x-3} \\ 5x-7 \end{array}$$

Area of a region, $A = \int_a^b f(x)\, dx$

Volume of a solid, $V = \pi \int_c^d [f(x)]^2 dx$

Topic 3: First Order Differential Equation

Order, Degree of a differential eqn

$$\left(\frac{d^2y}{dx^2}\right)^4 + 2\left(\frac{dy}{dx}\right)^7 - 5y = 3$$

order 2, degree 4

order of highest degree.

Separable Variable

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

variables can algebraically separated

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

First order linear differential eqn

$$\frac{dy}{dx} + P(x)y = Q(x)$$

variables are non-separable

① Determine an integrating factor, $v(x)$

$$v(x) = e^{\int P(x)\, dx}$$

② Multiply the differential eqn with $v(x)$

$$\boxed{v(x)\frac{dy}{dx} + P(x)v(x)y} = v(x)\,Q(x)$$

↓

$$\boxed{\frac{d\,[v(x)\,y]}{dx}} = v(x)\,Q(x)$$

③ Integrate with respect to x

$$v(x)y = \int v(x) \cdot Q(x)\, dx$$

Population Growth model

$$\frac{dy}{dt} = ky$$

rate of change of population $\propto$ population

$$\int \frac{1}{y}\, dy = \int k\, dt$$

$$\ln y = kt + c$$

$$y = e^{kt} \times \boxed{e^c}$$

$e^c = A$

$$y = Ae^{kt}$$

Radioactive Decay Models

$$\frac{dC}{dt} = -kC$$

decreasing

$$\int \frac{dC}{C} = \int -k\, dt$$

$$\ln C = -kt + c$$

$$C = e^{-kt} \times e^c$$

$$C = Ae^{-kt}$$

Newton's Law of Cooling

$$d\theta = -h(\theta - a)t$$

rate of cooling $\propto$ difference in temperature

$$\frac{d\theta}{\theta - a} = -h\, dt$$

$$\int \frac{d\theta}{\theta - a} = \int -h\, dt$$

$$\ln(\theta - a) = -ht + c$$

$$\theta - a = e^{-ht + c}$$

$$\theta = Ae^{-ht} + a$$

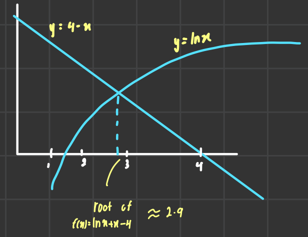
Topic 4: Numerical Method

Solution of non-linear eqn

① Graphical Method

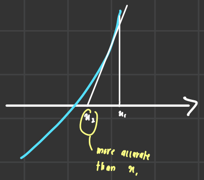
$\ln x + x - 4 = 0$

$y = \ln x$ $\quad$ $y = 4 - x$

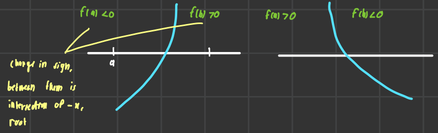


Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

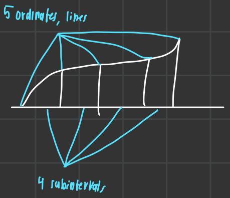


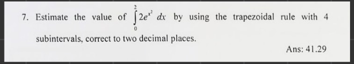
② Numerical Method



Trapezoidal Rule

if the value of $\int_a^b f(x)\,dx$ can't be integrated.





7. Estimate the value of $\int_0^2 2e^{x^2}\,dx$ by using the trapezoidal rule with 4 subintervals, correct to two decimal places.

Ans: 41.29

$h = \frac{2-0}{4}$

$= 0.5$

x	$y = 2e^{x^2}$	
0	2	
0.5		2.57
1.0		5.44
1.5		18.98
2.0	109.20	
	111.20	26.99

$A = \frac{0.5}{2}(111.20 + 2(26.99))$

$= 41.295 \text{ unit}^2$

Topic 6 : Permutations and Combinations

$0! = 1$

Permutations

$${}^4P_3 = \frac{4!}{(4\cdot3)!}$$ — arrange

Arrange PROBABILITY

$$\frac{11!}{2!2!}$$

Combinations

— choose

Topic 7: Probability

Random Experiment
↳ a process leading to ≥ 2 outcomes with uncertainty result

Basic Outcomes
↳ possible outcomes for random experiment

Sample space $S = \{1, 2, 3, 4, 5\}$
↳ set of all basic outcomes

Event
↳ subset of sample space

A : even number = {2, 4, 6}
B : odd number = {1, 3, 5}

Probablity of Events $\dfrac{n(A)}{n(S)} = \dfrac{\text{number of possible outcomes in } A}{\text{number of possible outcomes in sample space}}$

Complimentary Events, $P(\bar{A}) = 1 - P(A)$
↳ Probabity of A does not occur

Additive Rule of Probability, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability of three events, $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Mutually exclusive events

$P(A \cup B) = P(A) + P(B)$

$P(A \cap B) = 0$

Contingency tables

	A	A'	Total (Row)
B	$P(A \cap B)$	$P(A' \cap B)$	$P(B)$
B'	$P(A \cap B')$	$P(A' \cap B')$	$P(B')$
Total (column)	$P(A)$	$P(A')$	$P(A) + P(A')$ or $P(B) + P(B')$

Conditional Probability

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

B comes first

Read as "Probability of A given B)

Independent Events
↳ event A and B are independent

$P(A \mid B) = P(A)$

$P(B \mid A) = P(B)$

$P(A \cap B) = P(A) \cdot P(B)$

Topic 8 : Random Variables

Random Variables

Discrete Random Variable

can be counted, no interval

countable number of values

Properties:

1. $0 \leq P(X=x) \leq 1$
2. $\sum_{i=1}^{k} P(X=x_i) = 1$

summation of all probablity

Cumulative distribution function

$F(x) = \sum_{x_i}^{x} P(X=x)$ – summation of $P(x)$ from x upto X

Probablity distribution func of a random variable X

x	0	1	2	3
$p(x=x)$	1/8	3/8	3/8	1/8

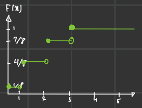
+

x	0	1	2	3
$F(x)$	1/8	4/8	7/8	1

Cumulative distribution func, $F(x)$.

$$F(x) = \begin{cases} 0, & x<0 \\ 1/8, & 0 \leq x < 1 \\ 4/8, & 1 \leq x < 2 \\ 7/8, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

graph F



i) Expectation of X

mean of X, = μ or $E(X)$

x	0	1	2	3
$p(x=x)$	1/8	3/8	3/8	1/8

$E(X) = 0(1/8) + 1(3/8) + 2(3/8) + 3(1/8)$

$=$

ii) Expectation of any function of X

x	1	2	3
$p(x=x)$	1/6	2/6	3/6

a) $E(2) = 2$

b) $E(X) = 1(\frac{1}{6}) + 2(\frac{2}{6}) + 3(\frac{3}{6})$
$= 7/3$

c) $E(5X) = 5(E(x)) = 5(7/3)$

d) $E(5X+2) = 5(E(x)) + 2$
$= 5(7/3) + 2$

iii) Variance of X

$Var(x) = E(X^2) - [E(X)]^2$

standard deviation $= \sqrt{Var(X)}$

Properties:

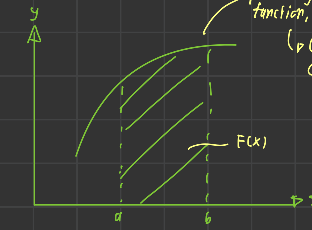
① $Var(a) = 0$

② $Var(aX) = a^2 Var(X)$

③ $Var(aX+b) = a^2 Var(X)$

Continuous Random Variables

uncountable values, x have interval



probablity density function, $f(x)$

① $f(x) \geq 0$

② $\int_{-\infty}^{\infty} f(x)dx = 1$

e.g 8.15

The continuous random variable X has probability density function

$$f(x) = \begin{cases} k(x+1), & 0 \leq x < 2 \\ 2k, & 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

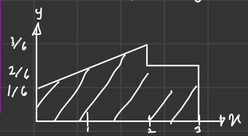
a) Value of constant k

$\int_0^2 k(x+1)\,dx + \int_2^3 2k\,dx = 1$

$k\left[\frac{x^2}{2} + x\right]_0^2 + 2k[x]_2^3 = 1$

$k = \frac{1}{6}$

b) sketch the probablity density function



c) Find $P(X > 1.8)$

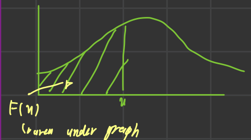
$P(X>1.8) = 1 - P(X \leq 1.8)$

$= 1 - \frac{1}{6}\int_0^{1.8} (x+1)dx$

$= 0.43$

Cumulative Distribution Func for Continuous Random Variable

$F(x) = \int_{-\infty}^{x} f(x)\,dx$



area under graph

e.g 8.17 : X is a continuous random variable with probablity density function $f(x) = \frac{x}{8}$, $0 \leq x \leq 4$

a) Find $F(x)$

for $x<0$, $F(x) = 0$

for $0 \leq x < 4$, $F(x) = \int_0^x \frac{1}{8}x\,dx = \frac{x^2}{16}$

for $x > 4$, $F(x) = 1$

c) Calculate $P(2<x<3)$

$= F(3) - F(2) = \frac{5}{16}$

Expectation and Variance of Random Variables

a) $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

b) $E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

c) $Var(X) = E(X)^2 - [E(x)]^2$

Topic 9 : Special Probablity Distribution

1/06/2022

Binomial Distribution, $X - B(n, p)$

— outcomes — success / fail

n: no of trials

p: probablity of success

↳ only 2 outcomes

$P(X=x) = {}^nC_x \, p^x q^{n-x}$

mean, $E(X) = np$

variance, $Var(x) = npq$

Poisson Distribution, $X \sim P_o(\lambda)$

λ: mean

↳ when x is number of occurances / average in a spare/ time

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

mean, $E(X) = \lambda$

Variance, $Var(X) = \lambda$

* Can be used for approximation of Binomial when:

- $n \geq 20$, $p \leq 0.05$
- $n \geq 100$, $p \leq 0.10$

$\} \; p \to 0, \; n \to \infty$

- mean = np

* if $\lambda > 30$, approximate ↓

Normal distribution, $X \sim N(\mu, \sigma^2)$

μ: mean, σ^2: variance

-standard normal distribution, $Z \sim N(0,1)$

standardizing the variable x, $z = \dfrac{x-\mu}{\sigma}$

* Can be used for approximation of Binomial when:

- $n > 30$, $np > 5$, $nq > 5$
- $n > 10$, $p \to 0.5$

Continuity Corrections

a) $P(X \geq a) = P(X > a - 0.5)$

b) $P(X > a) = P(X > a + 0.5)$

c) $P(X \leq a) = P(X < a + 0.5)$

d) $P(X < a) = P(X < a - 0.5)$

e) $P(a \leq X \leq b) = P(a - 0.5 < x < b + 0.5)$

f) $P(a < X < b) = P(a + 0.5 < x < b - 0.5)$

g) $P(a < X \leq b) = P(a + 0.5 < x < b + 0.5)$

h) $P(a \leq x < b) = P(a - 0.5 < x < b - 0.5)$

i) $P(X = a) = P(a - 0.5 < x < a + 0.5)$